Invariant Volume forms in Complex Geometry and their plurisubharmonic variations

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Abstract

In this report, I would like to explain several invariant volume forms in several complex variables and the pseudoconvex variation of them.

Contents

1 Convexity in Several Complex Variables

In several complex variables, the pseudoconvexity plays a central role. I would like to explain what is the pseudoconvexity and why it is important.

K. Oka proved the following fundamental theorem in 1953.

Theorem 1.1 *([O]) Let Ω be a domain in $\mathbb{C}^n$ and let $d_\Omega : \Omega \longrightarrow \mathbb{R}$ be the distance function , then the followings are equivalent.*

(1) Ω is a domain of holomorphy, i.e., there exists a holomorphic function $f \in \mathcal{O}(\Omega)$ which does not have an analytic continuation to a stictly larger domain.

(2) Ω is pseudoconvex, i.e,, $-\log d_\Omega$ is plurisubharmonic near the boundary, i.e,, $-i\partial\bar{\partial}\log d_\Omega$ is semipositive in the sense of current.

This theorem asserts that the natural existence domain of holomorphic functions has some **convexity**, i.e. "pseudoconvexity". The pseudoconvexity is the complex analytic counterpart of the geometric convexity.

For a domain in $\mathbb{C}$ with the smooth boundary, the difference of the geometric convexity and the pseudoconvexity is as follows.

Definition 1.1 *Let Ω be a domain in $\mathbb{C}^n$ with smooth boundary $\partial\Omega$.*

Ω is said to be geometrically convex, if there exists a local defining function of partialΩ such that the Hessian of the defininig function is positive definite on the boundary.

Ω is said to be pseudoconvex, if the local defining function of $\partial\Omega$ such that the complex Hessian of the defining fuction is positive definite on the boundary.

To generalize the notion of convexity for complex manifolds, we need to inroduce the notion of plurisubharmonic functions.

Definition 1.2 *Let Ω be a domain in $\mathbb{C}$ and let $\phi : \Omega \longrightarrow [-\infty, ; \infty)$ be an uppersemicontinuous function. ϕ is said to be* **subharmonic***, if for every $z \in \Omega$ and sufficiently small $r > 0$,*

$$\phi(z) \leqq \frac{1}{2\pi}\int_0^{2\pi} \phi(z + re^{i\theta})d\theta$$

holds.

It is well known $\phi \in C^2(\Omega)$ this ϕ is subharmonic, if and only if $\Delta\phi \geqq 0$ holds on Ω.

Now we shall introduce the notion of plurisubharmonic functions.

Definition 1.3 *Let Ω be a domain in $\mathbb{C}^n$, $\varphi : \Omega \longrightarrow [-\infty, \infty)$ is said to be* **plurisubharmonic** *, if for every $z \in \Omega$ and $w \in \mathbb{C}^n$, $\varphi(z+tw)$ is subharmonic in t for every sufficitnly small $|t|$.*

$\varphi \in C^2(\Omega)$ is plurisubharmonic. if and only if the complex Hessian $i\partial\bar{\partial}\varphi$ is positive semidefinite on Ω.

We denote the set of plurisubharmonic function on Ω by $PSH(\Omega)$. For $\varphi \in PSH(\Omega)$, $e^{-\varphi}$ is considered to be a hermitian metric on the trivial bundle $\Omega \times \mathbb{C}$ and its curvature is $\partial\bar{\partial}\varphi$ in the sense of current.

We define the intrinsic pseudoconvexity of a complex manifold as follows.

Definition 1.4 *Let X be a complex manifold. X is said to be pseudoconvex, if there exists a plurisubharmonic exhaustion function $\phi : X \longrightarrow \mathbb{R}$.*

Following the fundamental result of K. Oka, the basic philosophy of several complex variables is that every natural geometric object in several complex variables is pseudoconvex, in other words, is of (semi)positive curvature.

This is similar to the following fundamental theorem in algebraic geometry.

Theorem 1.2 *([Kod]) Let X be a compact complex manifold and let (L, h_L) be a hermitian line bundle with strictly positive curvature. Then*

$$H^q(X, K_X + L) = 0$$

holds for every $q \geqq 1$.

In fact K. Oka's fundamental theorem has proved in terms of the extension of Kodaira's work by Andreotti and Visentini([A-V]) and L. H́ormander ([H]).

2 Decomposition of geometry of compact Ḱahler manifolds

In this section, we shall discuss the decomposition of geometry of compact Ḱahler manifolds. This is called the minimal model program (MMP) in algebraic geometry.

2.1 Ḱahler manifolds

X : complex manifold
g : Hermitian metric on X. $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$
We define the fundametal 2-form ω by

$$\omega = \frac{\sqrt{-1}}{2} \sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

X; K$\acute{}$ahler $\Leftrightarrow d\omega = 0$

Hereafter we consider K$\acute{}$ahler manifolds.

The volume form of the n-dimensional K$\acute{}$ahler manifold (X, ω) is given by

$$dV(\omega) = \frac{\omega^n}{n!} = \det(g_{i\bar{j}})|dz_1 \wedge \cdots \wedge dz_n|^2$$

We define the Ricci form $\mathrm{Ric}(\omega)$ by

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

Let K_X denote the canonical bundle of X (the line bundle of $(n,0)$-forms on X. Then $\deg(g_{i\bar{j}}^{-1}$ defines a hermitian metric on K_X,i.e.,

$$\left(\frac{(\sqrt{-1})^{n^2}\sqrt{-1}\eta\wedge\bar{\eta}}{dV(\omega)}\right)^{\frac{1}{2}}$$

is a hermitian norm of η.

Hence $-\mathrm{Ric}(\omega)$ represents the de Rham cohomology class of the 1-st Chern class $c_1(K_X)$.

2.2 One dimensional case

In the case of one dimenisonal compact complex manifolds, it is classically known that every compact Riemann surface admits a K$\acute{}$ahler metric with constant curvature.

Let C be a Riemann surface, i.e., a connected complex manifold of dimension 1. Let $\pi : \tilde{C} \longrightarrow C$ be the universal covering. Then we have the following theorem:

Theorem 2.1 *(Koebe) $\tilde{C}$ is biholomorphic to $\mathbb{P}^1$, $\mathbb{C}$ or Δ, where Δ denotes the unit open disk in $\mathbb{C}$ (with) center O.*

This implies that every one dimensional complex manifold admits a complete K$\acute{}$ahler metric with constant curvature

$$\mathbb{P}^1 : \frac{\sqrt{-1}dz\wedge d\bar{z}}{(1+|z|^2)^2}$$

$$\mathbb{C} : \frac{\sqrt{-1}}{2}dz\wedge d\bar{z}$$

$$\Delta : \frac{\sqrt{-1}dz\wedge d\bar{z}}{(1-|z|^2)^2}$$

In particular we have the following theorem.

Theorem 2.2 *Let C be a compact Riemann surface of genus g. Then there exists a metric with constant (Ricci = sectional) curvature on C.*

This theorem follows from K$\acute{}$oebe's uniformization and the invariance of the Poincaré metric

$$\frac{\sqrt{-1}dz\wedge d\bar{z}}{(1-|z|^2)^2}$$

on Δ under the action of the automorphism group $\mathrm{Aut}(\Delta) \simeq \mathrm{SL}(2,\mathbb{R})$.

2.3 Calabi's conjecture

By Koebe's uniformization theorem (Theorem 2.1), every one dimensional complex manifold admits a metric with constant curvature.

But in the case of higher dimensional compact K$\acute{}$ahler manifolds. we cannot expect the existence of K$\acute{}$ahler metric with constant sectional curvature. In the case of higher dimensional compact K$\acute{}$ahler manifolds, the natural object to controle is the Ricci curvature.

Definition 2.1 *Let (X, ω) be a K$\acute{}$ahler manifold. X is said to be K$\acute{}$ahler-Einstein, if there exists a real number c such that*

$$\mathrm{Ric}(\omega) = c \cdot \omega$$

holds.

We say that $c_1(X)$ is positive, 0 or negative, if the de Rham chomology class $c_1(X)$ contains a K$\acute{}$ahler form, 0 or $-c_1(X)$ contains a K$\acute{}$ahler form

For the higher dimensional analogue of Koebe's uniformization theorem, the following conjecture is well known.

Conjecture 2.1 *(***Calabi's conjecture***) Let (X, ω) be a compact K"ahler manifolds,*

(1) If $c_1(X)$ is negative, then, there exists a unique K$\acute{}$ahler-Einstein form ω such that

$$-Ric(\omega) = \omega.$$

(2) For every volume form $d\mu$ on X, there exists a K$\acute{}$ahler form ω such that

$$d\mu = \frac{1}{n!}\omega^n$$

holds.

This conjecture has been proven by T, Aubin and S.T. Yau in (1978).[Au, Y1].

Theorem 2.3 *([Au], [Y1]) Let X be a compact K$\acute{}$ahler manifold with ample K_X. Then there exists a unique K$\acute{}$ahler form ω on X such that*

$$-\mathrm{Ric}(\omega) = \omega$$

holds.

Theorem 2.4 *([Y1]) Let X be a compact K$\acute{}$ahler manifold of dimension n and let ω be a K$\acute{}$ahler form on X and let $d\mu$ be a C^∞-volume form on X such that*

$$\int_X \omega^n = \int d\mu$$

holds. Then there exists a C^∞-function φ on X such that

$$(\omega + i\partial\bar{\partial}\varphi)^n = d\mu$$

holds.

In the case of $c_1(X) > 0$, there is a necessary and sufficient condition for the existence of K$\acute{}$ahler-Einstein form (Chen-Donaldson-Sung). But this result is rather complicated.

2.4 M.M.P (minimal model program)

The minimal model program (MMP) is a program to describe the geometry of projective algebraic varieties in terms of the following 3-classes of varieties:

(1) projective varieties with $c_1 > 0$ (Fano varieties),

(2) projective varieties with $c_1 = 0$ (Calabi-Yau varieties or Holomorphic symplective varieties),

(3) projective varieties with $c_1 < 0$ (projective varieties of general type).

These are the direct generalization of the case of compact Riemann surfaces.

Definition 2.2 (Fano manifold(Ricci positive part) *Let X be a smooth projective variety. X is said to be Fano manifold, if $-K_X$ is ample, i.e., $c_1(X)$ is positive.*

By Theorem2.4, every smooth Fano manifold admits a C^∞-Ḱa]hler metric with positive Ricci curvature.

Definition 2.3 (Ricci flat part)*Let X be a smooth projective variety. X is said to be varieties with trivial canonical classs, if K_X is numerically trivial, i.e., $c_1(X) = 0$.*

Definition 2.4 (Ricci negative part) *Let X be a smooth projective variety. X is said to be canonical model, if K_X is ample, i.e., K_X is positive.*

By Theorem2.4, every smooth Fano manifold admits a C^∞ Ricci flat Ḱa]hler metric.

This definition can be generalized to the case of varieties with mild singularities (i.e. with **canonical singularities**).

Definition 2.5 *Let X be a normal projective variety and let X_{reg} be the regular locus of X. Let $i : X_{reg} \longrightarrow X$ be the natural inclusion. Let $K_X := i_*\mathcal{O}_X(K_{X_{reg}})$.*

In the case of higher dimensional projective varieties, the geometry would be decomposed into 3-parts, in terms of minimal model program. This is a straightforward generalization of geometry of Riemann surfaces.

Definition 2.6 *A line bundle L on a compact Ḱahler manifold is said to be* **pseudoeffective (p.e.)***, if $c_1(L)$ sits on the closure of effetive cone (the cone of the classes of closed positive currents). A line bundle L is said to be* **nef(numerically effetive)***, if $c_1(L)$ sits on the closure of the Ḱahler cone of X.*

There is a conjecture in algebraic geometry similar to Calabi's conjecture (Conjecture **??**) in differential geomety,

Conjecture 2.2 *(***Minimal Model Conjecture***) Let X be a smooth projective variety. Then one of the followings holds.*

(1) There exists a fibration $f : X \longrightarrow Y$ such that a general fiber is a Fano variety (with positive dimension).

(2) *There exists a projective variety X_{min} birationally equivalent to X such that X is of canonical singularities and $K_X \in Div(X) \otimes \mathbb{Q}$ is nef.*

Definition 2.7 *Let X be a projective variety with only canonical singularities. X is said to be minimal, if K_X is nef.*

The minimal model conjecture asserts that we may single out the **''Fano part''** of a projective variety can be single out as a fiber of the fibration or the excepsional sets of the modification. The minimal model conjecture has been solved in the case of projective varieties of dimension $\leqq 4$. But in general it is still open.

The next step is to single out the geometry of varieties with trivial canonical class, i.e., varieties with $c_1 = 0$.

2.5 Iitaka fibration and Abundance conjecture

To single out the "Ricci flat part" of a compact K'ahler manifold, it is

Definition 2.8 *: Let X be a smooth projective variety and let K_X be the canonical line bundle on X. We define the* **Kodaira dimension** *$\kappa(X)$ of X by*

$$\kappa(X) := \limsup_{m \longrightarrow \infty} \frac{\log h^0(X, mK_X)}{\log m}.$$

It is known that $\kappa(X)$ is one of $-\infty, 0, 1, \cdots, \dim X$.

If $\kappa(X) \geqq 0$, for $m >> 1$,, we have a rational fibration

$$\Phi_{|m!K_X|} : X - \cdots \rightarrow Y = \Phi_{|m!K_X|}(X)$$

with $\dim Y = \kappa(X)$. This is called the **Iitaka fibration**. The rational map $\Phi_{|m!K_X|}$ is defined by

$$\Phi_{|m!K_X|}(x) = [\sigma_0(x) : \cdots : \sigma_N(x)] \in \mathbb{P}^N$$

where $\{\sigma_0, \cdots, \sigma_N\}$ is a set of basis of $H^0(X, m!K_X)$. The rational map $\Phi_{|m!K_X|}$ is not defined on the **base locus**: $\mathrm{Bs}|m!K_X| = \{x \in X | \sigma_j(x) = 0, 0 \leqq j \leqq N\}$.

Theorem 2.5 *(Iitaka) For a sufficiently large m, Y does not depend on m. And a general fiber F of $f : X - \cdots \rightarrow Y$, $\kappa(F) = 0$ holds.*

The Iitaka fibration single out the $\kappa = 0$ part of X.

Definition 2.9 *Let X be a smooth projective variety, We define the* **numerical Kodaira dimension** *$\nu(X)$ of X by*

$$\nu(X) = \sup_A \limsup_{m \rightarrow \infty} \frac{\log h^0(X, A + mK_X)}{\log m}$$

where A runs all the ample line bundle on X.

Conjecture 2.3 *(abundance conjecture) Let X be a smooth projective variety. Then $\kappa(X) = \nu(X)$ holds.*

If the minimal model conjecture holds, then the abundance conjecture is equivalent to the following conjecture:

Conjecture 2.4 *Let X be a minimal projective algebraic variety. Then K_X is semiample, i.e., there exists a positive integer N such that for every $m \geqq N$, $|m!K_X|$ is base point free and gives a fibration*

$$\Phi_{|m!K_X|} : X \longrightarrow Y$$

such that a general fiber F is a minimal algebraic variety with $m!K_F$ is trivial.

The minimal model conjecture and the abundance conjecture are still open. But the basic philosophy of these conjectures is we may single out the Ricci positive part and the Ricci flat part of the projective variety by some algebro-geometric operations.

X

3 K'ahler Ricci flow

The K'ahler-Ricci flow is the differential geometric counterpart of the minimal model conjecture and the abundance conjecture.

3.1 K'ahler Ricci flow

Let (X, ω_0) be a compact K'ahler manifold. We consider the following evolution equation.

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) \quad \text{on } X \times [0, T), \tag{1}$$

$$\omega(0) = \omega_0 \quad \text{on } X \times \{0\},$$

where T is the maximal existence time for the C^∞ solution.

Then we see that the de Rham cohomology class $[\omega]$ of ω satisfies

$$[\omega] = [\omega_0] + 2\pi t \cdot c_1(K_X)$$

holds. Let $\mathcal{K}$ denote the K'ahler cone, i,e, the set of the K'ahler classes on X.

Then the following theorem holds.

Theorem 3.1 *T is given by*

$$T = \sup\{t | [\omega_0] + 2\pi t \cdot c_1(K_X) \in \mathcal{K}\}.$$

3.2 Singular K'ahler-Ricci flows

By Theorem 3.1, if K_X is not nef, then the maximal existence time of the smooth solution for (1) is finite. But if we allow some singularities on the solution ω, we may construct a long time singluar solution for the K'ahler-Ricci flows.

Theorem 3.2 *Suppose that K_X is pseudoeffective. Then there exsist closed positive currenet $\omega(t)$ and a discrete sequence $0 < T_0 < T_1 \cdots < T_m < \cdots$ and a sequence of nonempty Zariski open subsets $\{U_m\}$*

$$X = U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_m \supset \cdots$$

such that

(1) $\omega(t)$ *is* C^{∞} *on* $U_m \times [T_{m-1}, T_m)$,

(2) $\omega(t)$ *satisfies the evolution equation :*

$$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) \quad \text{on } U_m \times [T_{m-1}, T_m),$$

(3) $\omega(t)$ *is a closed positive current of minmal singularities in the class* $[\omega_0] + 2\pi t \cdot c_1(K_X)$.

Here the current of minimal singularities is in the following sense.

Definition 3.1 *Let* T *be a closed positive*$(1,1)$ *current on a compact complex manifold* X. *Let* η *be a closed* $C^{\infty} - (1,1)$ *form in* $T]$

$$T = \eta + i\partial\bar{\partial}\varphi$$

for some $\varphi \in L^1(X)$. *We say that* T *is of minimal singularity if* φ *has minimal singularity among the closed positive current* $(1,1)$*-current in the same class, in the sense that for any* $\tilde{\varphi} \in L^1(X)$ *such that* $\eta + i\partial\bar{\partial}\tilde{\varphi}$ *is closed positive, there exists a positive constant* C *such that* $\tilde{\varphi} \leqq \varphi + C$ *holds on* X.

The following theorem follows from the result of [L, p.26].

Theorem 3.3 *Let* X *be a compact K*$\acute{}$*ahler manifold and let* $c \in H^{1,1}(X, \mathbb{R})$ *be a pseudoeffective class. Then there exists a closed positive* $(1,1)$*-current* T *in* c *with minimal singularities.*

Theorem 3.2 can be considered as a differential geometric version of minimal model program with scaling.

3.3 Rough correspondence between KRF and MMP

Construction of minimal model corresponds the KRF and the abundance conjecture corresponds the study of the limit of KRF.

Theorem 3.4 *Let* X *be a smooth pojective variety with pseudoeffective* K_X. *Let* ω_0 *be a smooth K*$\acute{}$*ahler form on* X. *Let us consider the KRF:*

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t))$$

starting from ω_0. *Suppose that* K_X *is abundant, i.e.,* $\kappa(X) = \nu(X)$ *holds. Then the limit*

$$\omega_{\infty} := \lim_{t \longrightarrow \infty} \frac{1}{t}\omega(t)$$

exists a generically smooth semi-K$\acute{}$*ahler form on* X *which is the generalized K*$\acute{}$*ahler-Einstein form on* X.

4 Basic Problems

Conjecture 4.1 *(Invariance of plurigenera) Let $f : X \longrightarrow S$ be a smooth $K\acute{}ahler$ family of compact $K\acute{}ahler$ manifolds (with connected base S). We set $P_m(s) = \dim H^0(X_s, mK_{X_s})$, Then $P_m(s)$ is constant on S for every $m \geqq 1$.*

This conjecture is still open.

Conjecture 4.2 *Let $f : X \longrightarrow S$ be a smooth $K\acute{}ahler$ family of compact $K\acute{}ahler$ manifolds. Suppose that a fiber has a pseudoeffective canonical bundle. Then the relative canonical bundle $K_{X/S} := K_X \otimes f^*K_S^{-1}$ is also pseudoeffective*

4.1 Case of projective families

In the case of projective family, we can solve the conjecture as follows. Let $f : X \longrightarrow S$ be a smooth projective family.

Definition 4.1 *Let X be a smooth projective manifold with pseudoeffective canonical bundle. Let A be a sufficitnly ample line bundle on X and let h_A be a C^∞ hermitian metric on A. We set*

$$d\mu_m = \sup\{h_A^{\frac{1}{m}}|\sigma|^{\frac{2}{m}} | \sigma \in H^0(X, A + mK_X), \int_X h_A^{\frac{1}{m}} \cdot |\sigma|^{\frac{2}{m}} = 1\}$$

where sup *denotes the pointwise supremum. We set*

$$d\mu_{can} := (\limsup_{m\to\infty} d\mu_m)^*$$

where $(\ )^*$ denotes the uppersemicontinuous envelope. We call $d\mu_{can}$ the **supercanonical measure**.

Definition 4.2 *Let L be a holomorphic line bundle on a complex manifold X and h_0 is a C^∞-hermitian metric on L. h is called a* **singular hermitian metric** *on L, if there exists $\varphi \in L^1_{loc}(X)|$ such that*

$$h = e^{-\varphi} \cdot h.$$

We call φ the weight function of h with respect to h_0 and we define the ideal sheaf $\mathcal{J}(h)$ by

$$\mathcal{J}(h)(U) = \{f \in \mathcal{O}(U) | |f|^2 e^{-\varphi} \in L^1_{loc}(U)\}.$$

We call $\mathcal{J}(h)$ the multiplier ideal sheaf of h. We define the curvature current Θ_h by

$$\Theta_h = \Theta_{h_0} + \partial\bar{\partial}\varphi.$$

Theorem 4.1 *(Nadel's vanishing theorem[N]) Let (X, ω) be a compact $K\acute{}ahler$ manifold and let (L, h) be a singular hermitian line bundle on X such that the curvature current satisfies the inequality $i\Theta_h \geqq \varepsilon\omega$ for some positive number ε. Then the multiplier ideal sheaf $\mathcal{J}(h)$ is coherent on X and we have the vanishing :*

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(h)) = 0$$

holds for every $q \geqq 1$.

AZD (Analytic Zariki Decomposition)

Definition 4.3 *Let X be a compact complex manifold and let L be a holomorphic line bundle on X. A singular hermitian metric h on L is said to be an analytic Zariski decomposition(**AZD**), if the followings hold.*

1. *Θ_h is a closed positive current,*
2. *for every $m \geq 0$, the natural inclusion*

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{J}(h^m)) \to H^0(X, \mathcal{O}_X(mL))$$

is an isomorphim. $\square$

Lemma 4.1 *$d\mu_{can}^{-1}$ is an AZD of K_X, i.e., $-\mathrm{Ric}(d\mu_{can})$ is a closed (semi)positive current and*

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{J}(d\mu_{can}^{-m})) \simeq H^0(X, \mathcal{O}_X(mK_X))$$

holds for every $m \geqq 1$

Lemma 4.2 *Let $f : X \longrightarrow S$ be a smooth projective family such that a fiber X_0 has pseudo effective K_{X_0}. Then K_{X_s} is pseudoeffective for every $s \in S$.*

Let A be a sufficietly ample line bundle on X . Then shrinking S if necessary, we may prove that for every $m \geqq 1$. We see that the restriction map

$$H^0(X, \mathcal{O}_X(mK_{X/S} + A)) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(mK_{X_0} + A))$$

is surjective (by induction on m and the L^2-extension theorem below).

Theorem 4.2 *(L^2-extension theorem) Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ contained in $\{|z_n| < 1\}$. Let $\varphi \in PSH(\Omega)$ and let $\Omega_0 = \Omega \cap \{z_n = 0\}$. We set*

$$A^2(\Omega, \varphi) = \{F \in \mathcal{O}(\Omega) | \int_{\Omega} |F|^2 e^{-\varphi} < \infty\}$$

and

$$A^2(\Omega_0, \varphi) = \{f \in \mathcal{O}(\Omega) | \int_{\Omega_0} |f|^2 e^{-\varphi} < \infty\}.$$

Then for every $f \in A^2(\Omega_0, \varphi)$, there exists $F \in A^2(\Omega, \varphi)$ such that $F|\Omega_0 = f$ and

$$\int_{\Omega} |F|^2 e^{-\varphi} \leqq \pi \int_{\Omega_0} |f|^2 e^{-\varphi}$$

This L^2-extension theorem can be generalized to the case of extension of twisted canonical forms.

Theorem 4.3 *Let $f : X \longrightarrow \Delta$ be a smooth projective family over a unit open disk Δ in $\mathbb{C}$. And let (L, h) be a singular hermitian line bundle on X with semipositive curvature current.*

Then the restriction map :

$$H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(h)) \longrightarrow H^0(X_0, \mathcal{O}_{X_0}(K_{X_0} + L|_{X_0}) \otimes \mathcal{J}(h_0))$$

is surjective.

4.2 Variation of supercanonical measures

In the projective case invariance of plurigenera can be proven as follows.

Theorem 4.4 *(Invariance of plurigenera[T5]) Let $f : X \longrightarrow S$ be a smooth projective family. Suppose that a fiber X_0 has pseudoeffective canonical bundle. Then X_s has pseudoeffective canonical bundle for every $s \in S$. We set*

$$d\mu_{X/S,can}|X_s = d\mu_{can,s}(s \in S)$$

Then $d\mu^{-1}_{X/S,can}$ is a singular hermitian metric on $K_{X/S}$ with semipositive curvature current. And $P_m(s) = h^0(X_s, mK_{X_s})$ is constant on S.

4.3 Basic strategy

To solve these conjecture, we shall use the following procedure:

Let $f : X \longrightarrow S$ be a smooth K'ahler family of compact K'ahler manifolds.

(1) Construct a family of invariant volume forms $d\mu_s$ for every $s \in S$ such that $d\mu_s^{-1}$ is an AZD of K_{X_s}.

(2) Define the relative volume form $d\mu_{X/S}$ by $d\mu_{X/S}|X_s = d\mu_s(s \in S)$ and prove that $-\mathrm{Ric}(d\mu_{X/S})$ is semipositive (in the sense of current).

(3) Apply the L^2-extension and prove the conjectures.

5 Bergman kernel

G: a domain in $\mathbb{C}^n$

$\mathcal{O}(G)$: the space of holomorphic functions on G

$\mathcal{O}(G) \cap L^2(G)$ is a Hilbert space by the inner product

$$(f, g) := \int_G f(z) \cdot \overline{g(z)}\, d\mu$$

where $d\mu$ denotes the standard Lebesgue measure on $\mathbb{C}^n$. But it is more convenient to consider the space of holomorphic n-forms instead of holomorphic functions. Instead of $f, g \in A^2(G)$, we consider the holomorphic n-form:

$$\sigma := f(z) \cdot dz_1 \wedge \cdots \wedge dz_n, \tau := g(z) \cdot dz_1 \wedge \cdots \wedge dz_n$$

Then we have

$$(\sqrt{-1})^{n^2} \int_G \sigma \wedge \bar{\tau} = 2^n (f, g)$$

holds. For a complex n-fold X, we denote K_X the holomorphic line bundle of $(n, 0)$-forms on X. K_X is said to be the canonical bundle of X.

G:a boudned domain in $\mathbb{C}^n$

$$A^2(G) := \left\{ \eta \in \Gamma(G, \mathcal{O}_G(K_G)) \,\middle|\, (\sqrt{-1})^{n^2} \int_G \eta \wedge \bar{\eta} < \infty \right\}.$$

$$(\eta, \eta') := (\sqrt{-1})^{n^2} \int_G \eta \wedge \overline{\eta'} \quad (\eta, \eta' \in A^2(G)).$$

with norm

$$\| \eta \| = (\eta, \eta)^{\frac{1}{2}}$$

Then $A^2(G)$ is a Hilbert space, i,e., it is complete space. This follows from the Montel's theorem and the (pluri)subharmonicity of the $|f|^2 (f \in \mathcal{O}(G))$.

$\{\phi_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of $A^2(G)$

$$K_G(z) = \sum_{i=1}^{\infty} |\phi_i(z)|^2 (\text{Bergman volume form})$$

where $|\phi(z)|^2 = (\sqrt{-1})^{n^2} \phi(z) \wedge \overline{\phi(z)}$.

$$\omega_G := \sqrt{-1}\partial\bar{\partial} \log K_G (\text{Bergman Kähler form})$$

The advantage of the Bergman volume form is that it is invaeriant under the action of holomorphic automorphism group $\mathrm{Aut}(G)$. This follows from the invariance of the inner product under that action of $\mathrm{Aut}(G)$.

ω_G is nothing but the pull-back of the Fubini-Study K'ahler form on $\mathbb{P}^{\infty}$ by the projective embedding

$$\Phi : G \longrightarrow \mathbb{P}^{\infty}$$

defined by

$$\Phi(z) = [\phi_1(z) : \phi_2(z) : \cdots : \phi_m(z) : \cdots]$$

By the invariance of $K_G(z)$ under the action of $\mathrm{Aut}(G)$, we see that ω_G is also invariant under the action of $\mathrm{Aut}(G)$.

5.1 Generalization of Bergman Kernels

Let Ω be a bounded domain in $\mathbb{C}^n$ and let $\varphi \in PSH(\Omega)$ be a plurisubharmonic function on Ω. We set

$$A^2(\Omega, \varphi) = \{\eta \in H^0(\Omega, \mathcal{O}(K_\Omega)) | \int_\Omega |\eta|^2 \cdot e^{-\varphi} < \infty\}.$$

Then $A^2(\Omega, \varphi)$ is a Hilbert space as before and we may define the weighted Bergman kernel $K(\Omega, \varphi)$ similarly as above.

Another generalization is the compact case. Let X be a compact complex manifold and let (L, h_L) be a (possibly singular) hermitian line bundle on X. Then we define the finite dimensional Hilbert space

$$A^2(X, K_X + L, h_L) := \{\sigma \in H^0(X, \mathcal{O}_X(K_X + L)) | \int_X |\sigma|^2 \cdot h_L < \infty\}.$$

Then we may define the Bergman kernel $K(X, K_X + L, h_L)(z, w)$ entirely as above.

5.2 Extremal Property of Bergman Kernels

Here we consider the diagonal part $K(\Omega, \varphi)$ and denote it as $K(\Omega, \varphi)(z) (z \in \Omega)$. The following property is called the extremal property of the Bergman kernel and it is very fundamental.

Proposition 5.1

$$K(\Omega,\varphi)(z) = \sup\{|\sigma|^2(z)| \parallel \sigma \parallel^2 = 1, \sigma \in A^2(\Omega,\varphi)\}$$

holds.

Using this property J.P. Demailly proved the following important approximation thoeorem.

Theorem 5.1 *([Dem]) Let $\varphi \in PSH(\Omega)$. Then*

$$\varphi = \lim_{m \longrightarrow \infty} \frac{1}{m} \log K(\Omega, m\varphi)$$

holds.

This theorem implies that every psh functions can be approximated by the logarithm of Bergman kernels.

5.3 Plurisubharmonic Variation of Bergman Kernels

Theorem 5.2 *(Maitani-Yamaguchi, Berndtsson) Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n \times \mathbb{C}$. Let $\varphi \in PSH(\Omega)$. Let $f : \Omega \longrightarrow \mathbb{C}$ be the second projection. Let $K(\Omega_t, \varphi_t)$ be the Bergman kernel of $\Omega_t = f^{-1}(t)$ with weight $\varphi_t := \varphi|\Omega_t$. Then ther relative voulme form K defined by*

$$K|_{\Omega_t} := K(\Omega_t, \varphi_t)$$

is PSH in the sense that $i\partial\bar{\partial} \log K \geqq 0$. holds on Ω.

The following theorem the projective version.

Theorem 5.3 *([B, B-P]) Let $f : X \longrightarrow Y$ be an algebraic fiber space and let (L, h_L) be a singular hermitian line bundle on X such that $\sqrt{-1}\Theta_{h_L} \geqq 0$. Then the singular hermitian metric h on $K_{X/Y} + L$ defined by*

$$h|X_y := K(X_y, K_{X_y} + L, h_L|X_y)^{-1} (y \in Y^\circ)$$

has semipositive curvature on X, where Y° denotes the complement of the discriminant locus of f. $\square$

6 Variation of K$\acute{}$ahler-Einstein metrics and Canonical measures

In this section, we shall investigate the plurisubharmonic variation properties of K$\acute{}$ahler-Einstein volume forms and canonical measures,

6.1 Polynomial Approximation of K'ahler-Einstein volume form

Let X be a smooth projective variety with ample canonical bundle of dimension n. Let A be a sufficiently ample line bundle on X and let h_A be a C^∞-hermitian metric on A. We set

$$K_1 := K(X, K_X + A, h_A).$$

Here every Bergman kernel is the diagonal part. For $m \geq 2$, inductively we define

$$K_m = K(X, mK_X + A, K_{m-1}^{-1})$$

Then we have the following theorem.

Theorem 6.1

$$dV_\infty = \lim_{m \longrightarrow \infty} h_A^{\frac{1}{m}} \sqrt[m]{(m!)^{-n} K_m}$$

is nothing but the K'ahler-Einstein volume form on X, i.e.,

$$dV_\infty = (2\pi)^n \frac{1}{n!} \omega_E^n$$

holds, where ω_E is the unique K-E form such that $-\mathrm{Ric}(\omega_E) = \omega_E$.

The above theorem asserts that we can approximate the K'ahler he following theorem is very important.

Theorem 6.2 *([T1]) Let $f : X \longrightarrow S$ be a smooth projective family with relatively ample canonical bundle. Let dV_s denote the unique K-E volume form on X_s. Then the relative volume form*

$$dV_{X/S}|X_s = dV_s$$

is semipositive in the sense that $-Ric\, dV_{X/S} \geqq 0$ holds.

This strategy can be generalized to the case of K'ahler-Ricci flow and gives the following theorem.

Theorem 6.3 *Let $f : X \longrightarrow S$ be a smooth projective family. Let A be an ample line bundle on X and let h_A be a C^∞-hermitian metric on A with positive curvature. Let $\omega_0 = i\Theta(h_A)$. We consider the family of K'ahler-Ricci flow*

$$\frac{\partial}{\partial t}\omega_s(t) = -\mathrm{Ric}(\omega_s(t))$$

$$\omega_s(0) = \omega_0|X_s.$$

Suppose that for some X_0, K_{X_0} is pseudoeffective. Then the family of KRF gives a fiberwise volume form $dV_s(t)$ such that

$$\omega_s(t) = -t \cdot \mathrm{Ric} dV_s(t) + \omega_0|X_s.$$

And the relative volume form

$$dV_{X/S}(t)|X_s = dV_s(t)$$

satisfies

$$-t\mathrm{Ric}\, dV_{X/S} + \omega_0$$

is semipositive current for every t.

Besides Bergman volume form, there are several invariant volume forms in complex geometry. Since there does not exist enough line bundle on a compact K'ahler manifold, in the case of K'ahler manifolds we need to consider the invariant volume forms other than Bergman volume forms. Here is a candidate:

Definition 6.1 *(Extremal measure) Let Ω be a bounded domain in $\mathbb{C}^n$. We set*

$$d\mu_{ext}(\Omega) = (\sup\{dV| - \mathrm{Ric}\, dV \geqq 0, \int_\Omega dV = 0\})^*$$

where dV runs uppersemicontinuous volume form on Ω. For a closed positive current T we define the twisted version:

$$d\mu_{ext}(\Omega, T) = (\sup\{dV| - \mathrm{Ric}\, dV + T \geqq 0, \int_\Omega dV = 0\})^*.$$

We also construct the extremal measure for an adjoint class on compact K'ahler manifolds.

$f : X \longrightarrow Y$: **algebraic fiber space**, i.e.,

- X, Y are smooth projective varieties.
- f is projective surjective morphism with connected fibers.
- $K_{X/Y} := K_X \otimes f^*K_Y^{-1}$: the relative canonical bundle.

7 Semipositivity of the direct image of pluricanonical systems

The following theorem is fundamental in algebraic geometry.

Theorem 7.1 *(Kawamata, 1982) If $\dim Y = 1$, then for every $m > 0$, $f_*K_{X/Y}^{\otimes m}$ is semipositive in the sense that every quotient $\mathcal{Q}$ of $f_*K_{X/Y}^{\otimes m}$, $\deg \mathcal{Q} \geqq 0$ holds.* □

The proof depends on the **variation of Hodge structure (VHS)** due to Griffiths and Schmid. The reason why we do not have the semipositive curvature property of $f_*K_{X/Y}^{\otimes m}$ is that the proof depends on the **Finslar metric** :

$$\parallel \sigma \parallel := \left(\int_{X/Y} |\sigma|^{\frac{2}{m}} \right)^{\frac{m}{2}}$$

on $f_*K_{X/Y}^{\otimes m}$.

7.1 Viehweg's weak semipositivity

E, Viehweg studied the direct image of pluricanonical systems by the fiber product method and find the weak semistability theorem..

Definition 7.1 *Let Y be a quasi-projective reduced scheme, $Y_0 \subseteq Y$ an open dense subscheme and let $\mathcal{G}$ be locally free sheaf on Y, of finite constant rank. Then $\mathcal{G}$ is* **weakly positive** *over Y_0, if for an ample invertible sheaf $\mathcal{H}$ on Y and for a given number $\alpha > 0$ there exists some $\beta > 0$ such that $S^{\alpha\cdot\beta}(\mathcal{G}) \otimes \mathcal{H}^\beta$ is globally generated over Y_0.* □

Definition 7.2 *Let $\mathcal{F}$ be a locally free sheaf and let $\mathcal{A}$ be an invertible sheaf, both on a quasi-projective reduced scheme Y. We denote*

$$\mathcal{F} \succeq \frac{b}{a} \cdot \mathcal{A},$$

if $S^a(\mathcal{F}) \otimes \mathcal{A}^{-b}$ is weakly positive over Y, where a, b are positive integers. □

Theorem 7.2 *([V1]) $f : X \longrightarrow Y$: an algebric fiber space such that $K_{X/Y}$ is* <u>*f-semiample*</u> *over the complement of the discriminant locus Y°.*

1. *(**Weak positivitiy**) $f_* K^m_{X/Y} (m > 0)$ is weakly positive over Y°.*
2. *(**Weak semistability**) There exists $e > 0$ such that*

$$f_* K^{\otimes m}_{X/Y} \succeq \frac{1}{e \cdot r(m)} \cdot \det(f_* K^{\otimes m}_{X/Y}) \quad \textit{on } Y^\circ. \ \square$$

- Y° : complement of the discriminant locus of f. $X^\circ := f^{-1}(Y^\circ)$.
- $r = \operatorname{rank} f_* \mathcal{O}_X(mK_{X/Y})$, $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiber product over Y
- $f^r : X^r \longrightarrow Y$ be the natural morphism.

$$\det f_* \mathcal{O}_X(mK_{X/Y}) \to \otimes^r f_* \mathcal{O}_X(mK_{X/Y}) = f^r_* \mathcal{O}_{X^r}(mK_{X^r/Y}). \tag{2}$$

Hence we have the canonical global section

$$\gamma \in \Gamma\left(X, f^{r*}(\det f_* \mathcal{O}_X(mK_{X/Y}))^{-1} \otimes \mathcal{O}_{X^r}(mK_{X/Y})\right). \tag{3}$$

Let Γ denote the zero divisor of γ. It is clear the Γ does not contain any fiber over Y°. Now we set

$$\delta_0 := \sup\{\delta > 0 | (X^r_y, \delta \cdot \Gamma_y) \text{ is KLT for every } y \in Y^\circ\}. \tag{4}$$

Theorem 7.3 *Let $f : X \longrightarrow C$ be an effectively parametrized family of canonically polarized varieties over a smooth quasiprojective curve C. Let m be a positive integer such that $f_* K^{\otimes m}_{X/C} \neq 0$. Then*

$$\deg \det f_* K^{\otimes m}_{X/C} \leqq \frac{1}{r\varepsilon} \deg K_C$$

holds, where ε is the threshold of $f_* K^{\otimes m}_{X/C}$ as in Theorem 8.1 and $r := \operatorname{rank} f_* K^{\otimes m}_{X/C}$.
□

In the case of the higher dimensional base, we have the following result.

Theorem 7.4 *Let $f : X \longrightarrow Y$ be an effectively parametrized family of canonically polarized varieties over a smooth quasiprojective curve C. Let m be a positive integer such that $f_* K^{\otimes m}_{X/Y} \neq 0$. Then*

$$\det f_* K^{\otimes m}_{X/Y} - \frac{1}{r\varepsilon} \deg K_Y$$

is not pseudoeffective, where ε is the threshold of $f_* K^{\otimes m}_{X/C}$ as in Theorem 8.1 and $r := \operatorname{rank} f_* K^{\otimes m}_{X/Y}$. □

Theorem 7.5 *(***Weak Semistabitlity Theorem** *[V2]) $f : X \longrightarrow Y$: algebraic fiber space and let Y° be the complement of the discriminant locus.*

1. *(***Semipositivity***) For $m > 0$ such that $f_*K_{X/Y}^{\otimes m} \neq 0$, then $f_*K_{X/Y}^{\otimes m}$ is Griffith semipositive with respect to the relative canonical measure.*
2. *(***Weak semistability***) There exist $e > 0$ and a singular hermitian metric $H_{m,e}$ on*
$$K_{X/Y}^{\otimes m} \otimes (f^* \det f_* K_{X/Y}^{\otimes m})^{-e}$$
with semipositive curvature current such that for every $y \in Y^\circ$ $H_{m,e}|X_y$ is an AZD of $K_{X/Y}^{\otimes m} \otimes (f^* \det f_* K_{X/Y}^{\otimes m})^{-e}|X_y$. □

7.2 KLT version

Theorem 7.6 *Let $f : X \longrightarrow Y$ be an algebraic fiber space and let D be an effective $\mathbb{Q}$ divisor on X such that (X, D) is KLT. Let Y° denote the complement of the discriminant locus of f. We set*

$$Y_0 := \{y \in Y | y \in Y^\circ, (X_y, D_y) \textit{ is a KLT pair}\}$$

- *Let a be a minimal positive integer such that mD is Cartier. Then there exist a positive integers b and m_0 such that for every $m \geqq m_0$, $b|m$, $m(K_{X/Y} + D)$ is Cartier and $f_*\mathcal{O}_X(m(K_{X/Y} + D))$ is globally generated over Y_0.*
- Let r denote *rank* $f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor)$ and let $X^r := X \times_Y X \times_Y \cdots \times_Y X$ be the r-times fiber product over Y and let $f^r : X^r \longrightarrow Y$ be the natural morphism. And let D^r denote the divior on X^r defined by $D^r = \sum_{i=1}^r \pi_i^* D$, where $\pi_i : X^r \longrightarrow X$ denotes the projection: $X^r \ni (x_1, \cdots, x_n) \mapsto x_i \in X$.

  There exists a canonically defined effective divisor Γ (depending on m) on X^r which does not conatin any fiber $X_y^r (y \in Y^\circ)$ such that if we we define the number δ_0 by

  $$\delta_0 := \sup\{\delta \,|\, (X_y^r, D_y^r + \delta\Gamma_y) \text{ is KLT for all } y \in Y^\circ\},$$

  then for every $\varepsilon < \delta_0$

  $$f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor) \succeq \frac{m\varepsilon}{(1 + m\varepsilon)r} \cdot \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor)$$

  holds over Y_0.
- There exists a singular hermitian metric $H_{m,\varepsilon}$ on $(1+m\varepsilon)(K_{X^r/Y} + D^r) - \varepsilon \cdot f^* \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor)^{**}$ such that
  1. $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ holds on X in the sense of current.
  2. For every $y \in Y_0$, $H_{m,\varepsilon}|X_y^r$ is well defined and is an AZD of

  $$(1 + m\varepsilon)(K_{X^r/Y} + D^r) - \varepsilon \cdot (f^r)^* \det f_*\mathcal{O}_X(\lfloor m(K_{X/Y} + D)\rfloor)^{**}|X_y$$

  □

7.3 Canonical measure

(Generalized Kähler-Einstein metrics)

Let $f : X \longrightarrow Y$ be an Iitaka fibration such that $(f_*K_{X/Y}^{\otimes m!})^{**}$ is locally free on Y for some m (hence for every sufficiently large m), where $**$ denotes the double dual. We define the $\mathbb{Q}$-line bundle

$$L := \frac{1}{m!}\,(f_*K_{X/Y}^{\otimes m!})^{**}$$

on Y. We note that L is independent of a sufficiently large m. L carries the natural singular hermitian metric h_L defined by

$$h_L^{m!}(\sigma,\sigma) = \left(\int_{X/Y}|\sigma|^{\frac{2}{m!}}\right)^{m!}.$$

(L, h_L) : **Hodge $\mathbb{Q}$-line bundle**

Theorem 7.7 *(***Existence of canonical measures (Song-Tian, T-)***) In the above notations, there exists a unique singular hermitian metric on h_K on $K_Y + L$ and a nonempty Zariski open subset U in Y such that*

1. *h_K is an AZD of $K_Y + L$.*
2. *h_K is real analytic on U.*
3. *$\omega_Y = \sqrt{-1}\,\Theta_{h_K}$ is a Kähler form on U.*
4. *$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_L} = \omega_Y$ holds on U.* □

The above equation:

$$-\mathrm{Ric}_{\omega_Y} + \sqrt{-1}\,\Theta_{h_L} = \omega_Y \tag{5}$$

is similar to the Kähler-Einstein equation :

$$-\mathrm{Ric}_{\omega_Y} = \omega_Y.$$

The correction term $\sqrt{-1}\,\Theta_{h_L}$ represents the isomorphism :

$$R(X, K_X)^{(a)} = R(Y, K_Y + L)^{(a)}$$

for some positive integer a, where for a graded ring $R := \oplus_{i=0}^{\infty}R_i$, where for a graded ring $R := \oplus_{i=0}^{\infty}R_i$ and a positive integer b, we set

$$R^{(b)} := \oplus_{i=0}^{\infty}R_{bi}.$$

We set

$$d\mu_{GKE} := f^*\left(\frac{\omega_Y^n}{n!}\cdot h_L^{-1}\right)$$

is called the canonical measure on X. $d\mu_{can}$ has the following properties.

- $d\mu GKE$ is a bounded volume form on X which degenerates along subvarieties on X.
- $d\mu_{GKE}^{-1}$ is an AZD of K_X.
- $d\mu_{GKE}$ is unique and birationally invariant.

7.4 Relative canonical measure

Theorem 7.8 *([T9]) Let $f : X \longrightarrow S$ be a projective family such that X, S are smooth and f has connected fibers. And let D be an effective divisor on X such that (X, D) is KLT. Suppose that $f_*\mathcal{O}_S\left(\lfloor m(K_{X/S} + D)\rfloor\right) \neq 0$ for some $m > 0$. Then there exists a singular hermitian metric h_K on $K_{X/Y} + D$ such that*

1. *Let us define $\omega_{X/S} := \sqrt{-1}\,\Theta_{h_K}$. Then $\omega_{X/S} \geqq 0$ holds on X.*
2. *For a general smooth fiber $X_s := f^{-1}(s)$ such that (X_s, D_s) is KLT, $h_K|X_s$ is $d\mu^{-1}_{can,(X_s,D_s)}$, where $d\mu_{can,(X_s,D_s)}$ denotes the canonical measure on (X_s, D_s). In particular $\omega_{X/S}|X_s$ is the canonical semipositive current on (X_s, D_s) constructed as in Theorem 8.1.*$\square$

7.5 Dynamical construction of the canonical measure

To prove the plurisubharmonic variation of canonical measure, we need to use the dynamical systems of Bergman kernels.

- X: a complex manifold,
- (L, h_L) : a singular hermitian line bundle on X.
- Hilbert space:

  $$A^2(X, K_X + L) := \{\sigma \in \Gamma(X, \mathcal{O}_X(K_X + L)) | (\sqrt{-1})^{n^2} \int_X h_L \sigma \wedge \overline{\sigma} < +\infty\}$$

- inner product:
  $$(\sigma, \sigma') := (\sqrt{-1})^{n^2} \int_X h_L \cdot \sigma \wedge \bar{\sigma'}$$
- $\{\sigma_i\}$: a complete orthonormal basis of $A^2(X, K_X + L)$
- $K(X, K_X + L, h_L) = \sum_i |\sigma_i|^2$: Bergman kernel of $K_X + L$ with respect to h_L.
- $$K(X, K_X + L, h_L)(x) = \sup\{|\sigma|^2(x); \sigma \in A^2(X, K_X + L, h_L), \parallel \sigma \parallel = 1\}$$

We shall consider the following setting.

- $f : X \to Y$: Iitaka fibration
- (L, h_L): Hodge $\mathbb{Q}$-line bundle
- A: sufficiently ample line bundle on Y
- h_A: C^∞ hermitian metric on A
- a : least positive integer such that $aL \in \mathrm{Div}(Y)$

$$K_1 := \begin{cases} K(Y, K_Y + A, h_A), & \text{if } a > 1 \\ \\ K(Y, K_Y + A + L), h_A \cdot h_L), & \text{if } a = 1 \end{cases}$$

and $h_1 := 1/K_1$.
Inductively we define $\{K_m\}$ and $\{h_m\}$ by

$$K_m := \begin{cases} K(Y, mK_Y + \lfloor \frac{m}{a} \rfloor aL + A, h_{m-1}), & \text{if } a \not| m \\ \\ K(Y, m(K_Y + L) + A, h_{m-1} \cdot h_L^a), & \text{if } a | m \end{cases}$$

Theorem 7.9 *(**Dynamical construction**) Let X be a smooth projective variety of nonnegative Kodaira dimension and let $f : X \longrightarrow Y$ be the Iitaka fibration as above. Let m_0 and $\{h_m\}_{m \geq m_0}$ be the sequence of hermitian metrics as above and let n denote $\dim Y$. Then*

$$h_\infty := \liminf_{m \to \infty} \sqrt[m]{(m!)^n \cdot h_m}$$

is a singular hermitian metric on $K_Y + L$ such that

$$\omega_Y = \sqrt{-1}\, \Theta_{h_\infty}$$

holds, where ω_Y is the canonical Kähler current on Y as in Theorem 8.1 and $n = \dim Y$.

In particular $\omega_Y = \sqrt{-1}\Theta_{h_\infty}$ (in fact h_∞) is unique and is independent of the choice of A and h_A. □

By Theorem 7.9, the theorem follows from the following theorem.

Theorem 7.10 *([B]) Let $f : X \longrightarrow Y$ be an algebraic fiber space and let (L, h_L) be a singular hermitian line bundle on X such that $\sqrt{-1}\Theta_{h_L} \geqq 0$. Then the singular hermitian metric h on $K_{X/Y} + L$ defined by*

$$h|X_y := K(X_y, K_{X_y} + L, h_L|X_y)^{-1} (y \in Y^\circ)$$

has semipositive curvature on X, where Y° denotes the complement of the discriminant locus of f.

8 Global generation

The following theorem is an important application of Theorem 7.9,

Theorem 8.1 *([T9]) Let $f : X \longrightarrow S$ be an algebraic fiber space. Then for every sufficiently large m, $f_*\mathcal{O}_X(m!K_{X/Y}))$ is globally generated on a fixed nonempty Zariski open subset S_0 of S.*

8.1 Scheme of the proof of Theorem 8.1

The scheme of the proof of Theorem 8.1 is summerized as follows:

- Plurisubharmonic variation of Canonical measures.
- Two Monge-Ampère foliations on the relative Iitaka fibrations and the base spaces induced by the $-\mathrm{Ric}$ of the relative canonical measure.
- Comparison of the two Monge-Ampère foliation in terms of the weak semistability
- Metrized canonical models are locally trivial along the leaves on the base.
- Leaves are closed and are the fibers of the moduli map to the moduli of metrized canonical models.
- The family of canonical measures defines a positive $\mathbb{Q}$-line bundle on the moduli space of the metrized canonical models.

8.2 Relative Iitaka fibration

$f : X \longrightarrow Y$ be an algebraic fiber space such that $\mathrm{Kod}(X/Y) \geqq 0$. Let Z be the image of the relative pluricanonical map

$$\Phi : X - \cdots \longrightarrow \mathbb{P}(f_*K_{X/Y}^{\otimes m!})$$

for $m >> 1$.

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & Z \arrow[dl, "h"] \\ & Y & \end{tikzcd}$$

For a sufficiently large m we see that a general fiber F of $g : X - \cdots \rightarrow Z$ is connected and $\mathrm{Kod}(F) = 0$. We call $g : X - \cdots \rightarrow Z$ the relative Iitaka fibration. By taking a suitable modification of X, we may assume that g is a morphism.

Let $f : X \longrightarrow Y$ be an algebraic fiber space and let $g : X \longrightarrow Z$ be a relative Iitaka fibration associated with $f_*K_{X/Y}^{\otimes m!}$. Taking a suitable modification we may and do assume the followings :

- g is a morphism,
- Z is smooth.
- $(g_*K_{X/Z}^{\otimes m!})^{**}$ is a line bundle on Z.

Let $h : Z \longrightarrow Y$ be the natural morphism.

8.3 Regularity of relative canonical measure

Let $f : X \rightarrow Y$ be an algebraic fiber space such that
$\mathrm{Kod}(X/Y) \geqq 0$and let $g : X \rightarrow Z$ be the relative Iitaka fibration as above.

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & Z \arrow[dl, "h"] \\ & Y & \end{tikzcd}$$

By the dynamical construction and the generalized Kähler-Einstein equation,we have the following lemma.

Lemma 8.1 *Let $d\mu_{X/Y,can}$ is C^ω on a Zariski open subset of X. Also the relative canonical Kähler current $\omega_{Z/Y}$ is C^ω on a Zariski open subset of Z.* $\square$

This can be proven by using the Dirichlet problem for complex Monge-Ampère equations and implicit function thoerems([C-K-N-S, Bl]).

8.4 Monge-Ampère foliation

The following theorem is very important and classical.

Theorem 8.2 *([B-K]) Let Ω: domain in $\mathbb{C}^n$ and let $f \in C^3(\Omega)$ be a plurisubharmonic function such that $dd^c f$ has constant rank say r on Ω. Then*

$$\mathcal{F} := \{\xi \in T\Omega | dd^c f(\xi, \bar{\xi}) = 0\}$$

defines a foliation on Ω such that the leaves are complex submanifolds of dimension $n - r$.

This foliation $\mathcal{F}$ is said to be a Monge-Ampère foliation on Ω associated with $dd^c f$.

Now we shall compare the two Monge-Ampere foliations.

$\omega_{Z/Y}$ defines a Monge-Ampère foliation $\mathcal{F}_Z$ on the generic point of Z. Let us consider the singular hermitian line bundle $(\det f_* K_{X/Y}^{\otimes m!}, \det h_m)$, where

$$h_m(\sigma, \sigma') := \int_{X/Y} \sigma \cdot \overline{\sigma'} \cdot d\mu_{X/Y,can}^{-(m!-1)}.$$

$\Theta_{\det h_m}$ defines a Monge-Ampère foliation $\mathcal{F}_Y$ on Y on the generic point of Y. The following is the key observation.

Lemma 8.2 *$h_* \mathcal{F}_Z = \mathcal{F}_Y$ holds.* $\square$

8.5 Weak stability

Now we shall use the fiber product technique of [**?**].

- $f : X \longrightarrow Y$: algebraic fiber space with $\mathrm{Kod}(X/Y) \geqq 0$.
- $r :=$ *rank* $f_* \mathcal{O}_X(mK_{X/Y})$.
- $X^r := X \times_Y X \times_Y \cdots \times_Y X$ (r-times),
- $f^r : X^r \longrightarrow Y$:the natural morphism.
- $f^r_* K_{X^r/Y}^{\otimes m} \simeq \otimes^r f_* K_{X/Y}^{\otimes m}$
- $\Gamma \in |K_{X^r/Y}^{\otimes m} \otimes (f^{r*} \det f_* K_{X/Y}^{\otimes m})^{-1}|$: corresponding to the inclusion :

  $(f^r)^*(\det f_* \mathcal{O}_X(mK_{X/Y})) \hookrightarrow (f^r)^* f^r_* \mathcal{O}_{X^r}(mK_{X^r/Y}) \hookrightarrow \mathcal{O}_{X^r}(mK_{X^r/Y})$.

- $$\delta_0 := \sup\{\delta \,|\, (X^r_y, \delta \cdot \Gamma_y) \text{ is KLT for all } y \in Y^\circ\},$$

- For every $\varepsilon < \delta_0$

$$f_*\mathcal{O}_X(mK_{X/Y}) \succeq \frac{m\varepsilon}{(1+m\varepsilon)r} \cdot \det f_*\mathcal{O}_X(mK_{X/Y})$$

  holds over Y°.

- There exists a singular hermitian metric $H_{m,\varepsilon}$ on $(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot (f^r)^* \det f_*\mathcal{O}_X(mK_{X/Y})^{**}$ such that $\sqrt{-1}\,\Theta_{H_{m,\varepsilon}} \geqq 0$ holds on X^r in the sense of current.

- For every $y \in Y^\circ$, $H_{m,\varepsilon}|X^r_y$ is well defined and is an AZD of

$$(1+m\varepsilon)K_{X^r/Y} - \varepsilon \cdot (f^r)^* \det f_*\mathcal{O}_X(mK_{X/Y})^{**}|X_y.$$

- Weak semistability $\Rightarrow \Theta_{h^*\det h_m}|\mathcal{F}_Z \equiv 0$
  Since $h_K := (\omega^n_{Z/Y})^{-1} \cdot h_L$ is an AZD of $K_{Z/Y} + L$,

$$(\omega^n_{Z/Y})^{-1} \cdot h_L = O(H_{m,\varepsilon} \otimes (h^* \det h_m)^\varepsilon)$$

  holds. This implies that h_K **is more positive than** $(h^* \det h_m)^\varepsilon$**.**
  This implies the assertion.

- Along the leaves of $\mathcal{F}_Y$, $h : (Z,(L,h_L)) \to Y$ is locally trivial, This is because $\sqrt{-1}\,\Theta_{h_m} \geqq 0$ and $\mathrm{trace}\,\Theta_{h_m} \equiv 0$ on $\mathcal{F}_Y$. Hence

$$\Theta_{h_m} \equiv 0 \ \text{ along } \mathcal{F}_Y$$

  Then the parallel transport on $f_*K^{\otimes m}_{X/Y}$ trivialize $(Z,(L,h_L))$ along $\mathcal{F}_Y$.

Hence we have that
$f_*\mathcal{F}_Z = \mathcal{F}_Y$

8.6 Metrized canonical models

- (X, D): KLT pair with $\mathrm{Kod}(X, D) \geqq 0$.
- $R(X, K_X + D) := \oplus_{m=0}^\infty \Gamma(X, \mathcal{O}_X(\lfloor m(K_X + D)\rfloor))$: the log canonical ring of (X, D) : finitely generated.
- $Y := \mathrm{Proj}\, R(X, K_X + D)$: the canonical model of (X, D).
- $L := \frac{1}{m_0!}\left(f_*\mathcal{O}_X(m_0!(K_{X/Y} + D))\right)^{**}$ $(m_0 >> 1)$: the Hodge $\mathbb{Q}$-line bundle.
- h_L: the Hodge metric on L.
- ω_Y : the canonical Kähler current.
- $h_K := n!(\omega^n_Y)^{-1} \cdot h_D (n = \dim Y)$: canonical metric on $K_Y + L$.

Definition 8.1 *The pair* $(Y,(L,h_L))$ *is called the* **metrized canonical model** *associated with the KLT pair* (X, D)*.*

The moduli space of metrized canonical models

Let $(Z_y, (L, h_L)|Z_y)$ be the canonical model Z_y of X_y and the metrized Hodge bundle. The Hodge metric comes from a variation of Hodge structure on the canonical cyclic cover $W_y^\circ \longrightarrow Z_y^\circ$.

$$\mathcal{M} = \{(Z_y, (L, h_L)|Z_y)\}/\sim$$

where the equivalece $\sim$ is defined by

$$\varphi : Z_y \longrightarrow Z_{y'}$$

covered by the biholomorphism $\tilde{\varphi} : W_y^\circ \longrightarrow W_{y'}^\circ$ which induces an isomorphism between flat bundles preserving the Hodge line bundles.

Theorem 8.3 *$\mathcal{M}$ has a structure of separable complex space and for $m >> 1$ (some multiple of) $\det f_*\mathcal{O}_X(m!K_{X/Y})$ decends to a polarization of $\mathcal{M}$. In particular $\mathcal{M}$ is quasiprojective.* $\square$

This theorem implies that the leaves of $\mathcal{F}_Y$ is the fiber of the classifying map :

$$\Phi : Y^\circ \longrightarrow \mathcal{M}.$$

Then some symmetric power $S^r(f_*\mathcal{O}_X(m!K_{X/Y}))$ decends to a vector bundle on $\mathcal{M}$. Then by the **weak semistability**, we see that for $m >> 1$ $S^r(f_*\mathcal{O}_X(m!K_{X/Y}))$ decends to a very ample vector bundle on $\mathcal{M}$. Then $f_*\mathcal{O}_X(rm!K_{X/Y})$ is globally generated on Y° for $m >> 1$.

This theorem gives an alternative proof of the following theorem.

Theorem 8.4 *(Viehweg) Let $\mathcal{M}_{pol,min}$ be the polarized minimal algebraic varieties with semiample canonical divisors, then $\mathcal{M}_{pol,min}$ is quasiprojective .* $\square$

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